On Averaging for Hamiltonian Systems with One Fast Phase and Small Amplitudes.

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1 Introduction.

The problem of averaging for systems with one fast phase was considered from various points of view in many papers. The averaging method of Krylov and Bogolyubov [1] and methods of KAM theory originated this line of research, the most complete results were obtained by Neishtadt [2], where the coefficients are assumed real analytic. However, in many problems which are interesting from the point of view of applications, analytic dependence ceases to hold in the neighborhoods of some points.

For instance, consider the motion of a particle under the influence of a small periodic electric and constant magnetic field [3]. The Hamiltonian of such a system has the form :

$$H = \frac{1}{2}\left((p_1 + x_2)^2 + p_2^2\right) + \varepsilon V(x_1, x_2).$$

The canonical change of variables $x_1 = Q + y_1$, $p_1 = -y_2$, $x_2 = P + y_2$, $p_2 = -Q$, $Q = \sqrt{2I}\cos\varphi$, $P = \sqrt{2I}\sin\varphi$, transforms it into:

$$H = I + \varepsilon V\left(\sqrt{2I}\cos\varphi + y_1, \sqrt{2I}\sin\varphi + y_2\right).$$

This system depends on one fast phase φ and does not depend analytically on I near $I = 0$. This is not a minor problem since for many interesting problems coming from physics, a neighborhood of $I = 0$ may play the most important role. In the problem of semiclassical quantization, for examples it corresponds to the so-called low levels of Landau which are connected with Hall's conductivity. This problem is actually the main motivation of present paper.

The procedure in [2] is based upon subsequent change of variables, which corresponds to methods of KAM-theory. The aim of this paper is to show that one can choose a transformation such that the averaging procedure of [2] is applicable in a neighborhood of $I = 0$, and such that the passage from $I > \varkappa > 0$ to $I = 0$ is uniform.

2 Formulation of the problem and the main result.

Consider the Hamiltonian

$$H = \mathcal{H}_0\,(I) + \varepsilon g_0(q,p,y_1,y_2), \qquad\qquad I = \frac{q^2+p^2}{2}, \tag{1}$$

where $0 < \varepsilon < \varepsilon_0$ is a small parameter, and $\mathcal{H}_0$ and g_0 are real analytic functions in a complex δ-neighborhood of the domain $D := D_{2n}\{y_1,y_2\} \times D_2^{\varkappa}\{q,p\}$, $D_{2n} \subset \mathbb{R}^{2n}$, $D_2^{\varkappa} = \{(q,p) \in \mathbb{R}^2 \,|\, I < \varkappa\}$. In D we assume the following conditions

$$|\mathcal{H}_0| \le C, \qquad |g_0| \le C, \qquad \left|\frac{\partial \mathcal{H}_0}{\partial I}\right| \ne 0.$$

Following [1, 2], we show that for every integer $m > 0$ there exists a close to identity real analytic canonical transformation $(q,p,y_1,y_2) \to (Q,P,z_1,z_2)$ defined by

$$\begin{cases} q = Q + \varepsilon Q^1(Q,P,z_1,z_2,\varepsilon), & y_1 = z_1 + \varepsilon Z_1^1(Q,P,z_1,z_2,\varepsilon), \\ p = P + \varepsilon P^1(Q,P,z_1,z_2,\varepsilon), & y_2 = z_2 + \varepsilon Z_2^1(Q,P,z_1,z_2,\varepsilon), \end{cases} \tag{2}$$

where $|Q^1| + |P^1| + |Z_1^1| + |Z_2^1| \le C$, and such that the Hamiltonian (1) transforms into

$$H = \mathcal{H}_m\left(\frac{Q^2+P^2}{2}, z_1, z_2, \varepsilon\right) + \varepsilon g_m(Q,P,z_1,z_2,\varepsilon). \tag{3}$$

Precisely, we have the following theorem.

Theorem 1. Assume the conditions above in $(Q,P,z_1,z_2) \in D + \frac{1}{2}\delta$. Then there exists some interval $(0,\varepsilon_1]$, integer number r and real analytic canonical change of variables of the form(2), which transforms Hamiltonian (1) into (3) with exponentially small g:

$$|g_r| + |\nabla g_r| < c_2 \exp\left(-\frac{1}{c_1\varepsilon}\right), \qquad |Q^1| + |Z_2^1| + |P^1| + |Z_1^1| < c_3, \qquad |\mathcal{H}_r - \mathcal{H}_0| < c_4\varepsilon. \tag{4}$$

Here $\varepsilon \in (0,\varepsilon_1]$ and $\varepsilon_1, r, c_i, i = 1,2,3,4$ depend on ε_0, δ, C and $\varkappa$.

3 Auxiliary lemmas.

Let $w\,(I,\mu)$ and $g(q,p,\mu)$ be an analytic function of I and (q,p) respectively and vector-parameter μ , $\frac{\partial w}{\partial I} \ne 0$, $I = (q^2+p^2)/2$. Denote $\frac{\partial w}{\partial q} = q\frac{\partial w}{\partial I}$, $\frac{\partial w}{\partial p} = p\frac{\partial w}{\partial I}$. Consider the equation :

$$\frac{\partial w}{\partial p}\frac{\partial W}{\partial q} - \frac{\partial w}{\partial q}\frac{\partial W}{\partial p} + g(q,p,\mu) = \bar{g}\left(\frac{q^2+p^2}{2}, \mu\right), \tag{5}$$

$$\bar{g}\left(\frac{q^2+p^2}{2}, \mu\right) = \int_0^{2\pi} g\,(q(\varphi,I), p(\varphi,I), \mu)\, d\varphi\Big|_{\left(\substack{\varphi=\varphi(q,p)\\ I=I(q,p)}\right)}, \tag{6}$$

$$g(q,p,\mu) = \bar{g}\left(\frac{q^2+p^2}{2}, \mu\right) - \tilde{g}(q,p,\mu), \tag{7}$$

where $\bar{g}((q^2+p^2)/2,\mu)$ is a mean value of $g(q,p,\mu)$ with respect to φ, and the rest part of g is noted as $\tilde{g}(q,p,\mu)$ and it is convenient to take sign minus here.

Lemma 1. Equation (5) is solvable and has analytic solution on variables (q,p) and parameter μ. Function $W(q,p,\mu)$, defined by formula :

$$W(q,p,\mu) = \frac{1}{\frac{\partial w}{\partial I}} \left(\frac{1}{2} \int\limits_0^{\varphi} \tilde{g}(q(\psi,I),p(\psi,I),\mu) d\psi + \frac{1}{2} \int\limits_{\pi}^{\varphi} \tilde{g}(q(\psi,I),p(\psi,I),\mu) d\psi \right) \Bigg|_{\varphi=\varphi(q,p),I=I(q,p)}, \tag{8}$$

is an analytic solution of equation (5).

Remark. General solution of equation (5) has a form :

$$W(q,p,\mu) = \left(\frac{1}{\frac{\partial w}{\partial I}} \int\limits_0^{\varphi} \tilde{g}(q(\psi,I),p(\psi,I),\mu) d\psi + W_0(I,\mu) \right) \Bigg|_{\varphi=\varphi(q,p),I=I(q,p)},$$

where $W_0(I,\mu)$ is a constant of integration. It follows for (q,p) variables from method [2]. We show below, that in the neighborhood of point $q=0, p=0$ definition of constant of integration, pointed in (8), allow us to integrate equation (5), preserving analyticity of solutions.

Proof.

First make a canonical change of variables:

$$q = \frac{u+iv}{\sqrt{2}}, \qquad p = \frac{v+iu}{\sqrt{2}}.$$

Then equation (5) is :

$$\left(\frac{\partial w}{\partial v}\frac{\partial W}{\partial u} - \frac{\partial w}{\partial u}\frac{\partial W}{\partial v} \right) = \tilde{g}\left(\frac{u+iv}{\sqrt{2}}, \frac{v+iu}{\sqrt{2}}, \mu \right). \tag{9}$$

Now we should expand $\tilde{g}$ into Taylor series. The expansion has form:

$$\tilde{g}\left(\frac{u+iv}{\sqrt{2}}, \frac{v+iu}{\sqrt{2}}, \mu \right) = \sum_{k,l\in\mathbb{N}} \tilde{g}_{kl}(\mu) u^k v^l \frac{(k+l)!}{k!l!}.$$

Using coordinates (φ,ρ): $u=\rho e^{i\varphi}$, $v=\rho e^{-i\varphi}$ one can easily integrate equation (9):

$$W = \frac{\rho}{\frac{\partial w}{\partial \rho}} \left(\int\limits_0^{\varphi} \tilde{g}\left(\frac{u+iv}{\sqrt{2}}, \frac{v+iu}{\sqrt{2}}, \mu \right) d\psi \right) + W_0(\rho,\mu) =$$

$$= \frac{\rho}{\frac{\partial w}{\partial \rho}} \left(\sum_{k,l\in\mathbb{N}} \tilde{g}_{kl}(\mu) \frac{u^k(\varphi,\rho) v^l(\varphi,\rho)}{i(k-l)} \frac{(k+l)!}{k!l!} - \sum_{k,l\in\mathbb{N}} \tilde{g}_{kl}(\mu) \frac{|uv|^{(k+l)}}{i(k-l)} \frac{(k+l)!}{k!l!} \right) + W_0(\rho,\mu), \tag{10}$$

where $W_0(\rho,\mu)$ is a constant of integration.

We can see that at bottom limit of integration we can have nonanalytic ($\sim uv = \sqrt{2I}$) dependence on I at the point $(u = 0, v = 0)$. But we can choose $W_0(\rho, \mu)$ in a such way that nonanalytic term disappears. Introduce W_0 as :

$$W_0(\rho,\mu) = \frac{\rho}{\frac{\partial w}{\partial \rho}} \left(\frac{1}{2} \int\limits_{\pi}^{\varphi} \tilde{g}(u(\psi,\rho), v(\psi,\rho),\mu) d\psi - \frac{1}{2} \int\limits_{0}^{\varphi} \tilde{g}(u(\psi,\rho), v(\psi,\rho),\mu) d\psi \right) =$$

$$= \frac{\rho}{\frac{\partial w}{\partial \rho}} \left(\sum_{k,l\in\mathbb{N}} \tilde{g}_{kl}(\mu) \frac{|uv|^{(k+l)}}{i(k-l)} \frac{(k+l)!}{k!l!} \right). \tag{11}$$

Hence (11) 'kills' nonanalytic term in (10), so W is analytic function of I in the point $I = 0$. It is easy to see that $\frac{\partial w}{\rho \partial \rho} = \frac{\partial w}{\partial I}$. Function $W(q, p, \mu)$ is analytic ones of parameter μ due to procedure of construction: we had no nonanalytic dependence on μ on all steps of obtaining of solution. Hence, W is an analytic function of (q, p) and parameter μ. Lemma is proved.

We use generating function to construct canonical transformation in the proof of main theorem.

Let $S(q, P, y_1, z_2, \varepsilon)$ be an analytic function of all variables (q, P, y_1, z_2) and small parameter $\varepsilon \in [0, \varepsilon_0]$, $(q, p, y_1, y_2) \in U$, $(Q, P, z_1, z_2) \in U - \delta$, $\delta > 0$, $U \subset \mathbb{R}^{2n+2}$. Domain $U - \delta$ is the set of points from U, which enter together with their δ-neighborhood [5]. Consider a system of equations :

$$\begin{cases} Q = q + \varepsilon \dfrac{\partial S(q,P,y_1,z_2,\varepsilon)}{\partial P}, & z_1 = y_1 + \varepsilon \dfrac{\partial S(q,P,y_1,z_2,\varepsilon)}{\partial z_2}, \\ p = P + \varepsilon \dfrac{\partial S(q,P,y_1,z_2,\varepsilon)}{\partial q}, & y_2 = z_2 + \varepsilon \dfrac{\partial S(q,P,y_1,z_2,\varepsilon)}{\partial y_1}, \end{cases} \tag{12}$$

and assume that $\max\left\{ |\frac{\partial S}{\partial q}|, |\frac{\partial S}{\partial P}|, |\frac{\partial S}{\partial y_1}|, |\frac{\partial S}{\partial z_2}| \right\} < C_U$ in domain U.

Following lemma establishes that introduced transformation is a change of variables and allows us to estimate corrections.

Lemma 2. If $\varepsilon < \delta/(2C_U(n+1))$, where n is a dimension of y_1, then the system (12) is solvable and solution has the following form:

$$\begin{cases} q = Q + \varepsilon Q^1(Q,P,z_1,z_2,\varepsilon), & y_1 = z_1 + \varepsilon Z_1^1(Q,P,z_1,z_2,\varepsilon), \\ p = P + \varepsilon P^1(Q,P,z_1,z_2,\varepsilon), & y_2 = z_2 + \varepsilon Z_2^1(Q,P,z_1,z_2,\varepsilon), \end{cases} \tag{13}$$

where Q^1, P^1, Z_1^1, Z_2^1 are analytic functions in $U - \delta \times [0, \varepsilon_0]$ and $\max\{|Q^1|, |P^1|, |Z_1^1|, |Z_2^1|\} < C_U$ in domain $U - \delta$.

Proof.

Let us present system (12) as :

$$\begin{cases} f_1(q,p,y_1,y_2;Q,P,z_1,z_2,\varepsilon) = 0, & f_3(q,p,y_1,y_2;Q,P,z_1,z_2,\varepsilon) = 0, \\ f_2(q,p,y_1,y_2;Q,P,z_1,z_2,\varepsilon) = 0, & f_4(q,p,y_1,y_2;Q,P,z_1,z_2,\varepsilon) = 0, \end{cases} \tag{14}$$

and $F = (f_1, f_2, f_3, f_4)$. Due to the theorem of implicit function, the system (14) is solvable with respect to (Q, P, z_1, z_2), if $\det |\frac{DF}{D(q,p,y_1,y_2)}| \neq 0$, and its solution is analytic in $U - \delta \times [0, \varepsilon_0]$. In our case

$$\det \left| \frac{DF}{D(q,p,y_1,y_2)} \right| = \det \begin{vmatrix} 1 + \varepsilon \dfrac{\partial^2 S}{\partial P \partial q} & \varepsilon \dfrac{\partial^2 S}{\partial P \partial y_1} \\ \varepsilon \dfrac{\partial^2 S}{\partial z_2 \partial q} & 1 + \varepsilon \dfrac{\partial^2 S}{\partial z_2 \partial y_1} \end{vmatrix}.$$

$S(q, P, y_1, z_2, \varepsilon)$ is an analytic function of ε. Using Cauchy estimations for analytic functions [4], we obtain

$$\det \left| \frac{DF}{D(q,p,y_1,y_2)} \right| - 1 < \sum_{i=1}^{n+1} \left(\varepsilon \frac{C_U}{\delta} \right)^i \frac{(n+1)!}{(n+1-i)!} < \sum_{i=1}^{\infty} \left(\varepsilon \frac{C_U}{\delta} (n+1) \right)^i < 1. \qquad (15)$$

Hence, if $\varepsilon < \delta / (2C_U(n+1))$ then (15) is valid on U always and our system is solvable. Uniqueness of the transformation is established similarly to [5]. The solutions should satisfy the equations (12) with $\varepsilon = 0$ also, so they can be represented as (13). Let us substitute the solutions in equations (12). Then from equations for q and y_1 we have the following form for Q^1 and Z_1^1:

$$Q^1 = -\frac{\partial S(Q + \varepsilon Q^1, P, z_1 + \varepsilon Z_1^1, z_2, \varepsilon)}{\partial P}, \qquad Z_1^1 = -\frac{\partial S(Q + \varepsilon Q^1, P, z_1 + \varepsilon Z_1^1, z_2, \varepsilon)}{\partial z_2}.$$

Hence $|Q^1| < C_U$ and $|Z_1^1| < C_U$. For p and y_2 variables we have the following formulae:

$$p = P + \varepsilon \frac{\partial S(Q + \varepsilon Q^1, P, z_1 + \varepsilon Z_1^1, z_2, \varepsilon)}{\partial q} = P + \varepsilon \frac{\partial S(Q + \varepsilon Q^1, P, z_1 + \varepsilon Z_1^1, z_2, \varepsilon)}{\partial Q},$$
$$y_2 = z_2 + \varepsilon \frac{\partial S(Q + \varepsilon Q^1, P, z_1 + \varepsilon Z_1^1, z_2, \varepsilon)}{\partial y_1} = z_2 + \varepsilon \frac{\partial S(Q + \varepsilon Q^1, P, z_1 + \varepsilon Z_1^1, z_2, \varepsilon)}{\partial z_1}.$$

Therefore:

$$P^1 = \frac{\partial S(Q + \varepsilon Q^1, P, z_1 + \varepsilon Z_1^1, z_2, \varepsilon)}{\partial Q}, \qquad Z_2^1 = \frac{\partial S(Q + \varepsilon Q^1, P, z_1 + \varepsilon Z_1^1, z_2, \varepsilon)}{\partial z_1},$$

and again $|P^1| < C_U$, $|Z_2^1| < C_U$. Lemma is proved.

4 Proof of the theorem.

We construct our change of variables as a composition of large number of consequently defined canonical transformations, giving dependence on $(q^2 + p^2)/2$ of Hamiltonian more and more higher degree of ε.

4.1 Procedure of consequently defined changes of variables.

Assume that Hamiltonian is obtained after i changes of variables has the following form :

$$H = \mathcal{H}_i\left(\frac{q^2+p^2}{2}, y_1, y_2, \varepsilon\right) + \varepsilon g_i(q, p, y_1, y_2, \varepsilon), \tag{16}$$

$$(q, y_2, p, y_1) \in D_i, \qquad D_i = D_1 - 2(i-1)K\varepsilon,$$

$$D + \frac{\delta}{2} \subset D_i \subset D + \delta, \qquad D_1 = D + \frac{3}{4}\delta.$$

At $i+1$ step one has to find canonical infinitesimal ("almost identical") change of variable (see [2, 4, 5]) $(q, p, y_1, y_2) \to (Q, P, z_1, z_2)$

$$\begin{cases} q = Q + \varepsilon Q^1(Q, P, z_1, z_2, \varepsilon), & y_1 = z_1 + \varepsilon Z_1^1(Q, P, z_1, z_2, \varepsilon), \\ p = P + \varepsilon P^1(Q, P, z_1, z_2, \varepsilon), & y_2 = z_2 + \varepsilon Z_2^1(Q, P, z_1, z_2, \varepsilon), \end{cases} \tag{17}$$

such that Hamiltonian takes a form :

$$H = \mathcal{H}_{i+1}\left(\frac{Q^2+P^2}{2}, z_1, z_2, \varepsilon\right) + \varepsilon g_{i+1}(Q, P, z_1, z_2, \varepsilon), \tag{18}$$

$$\mathcal{H}_{i+1} = \mathcal{H}_i + \varepsilon\bar{g}_i, \qquad g_{i+1} = O(\varepsilon^{i+1}),$$

where $\mathcal{H}_{i+1}$ contains terms of order $i+1$ of ε, and the operation "bar" over g_i is defined in sec. 3.

It is possible and convenient to do by means of generating function (then one has the canonical property automatically) :

$$S = S(q, P, y_1, z_2, \varepsilon) = qP + y_1 z_2 + \varepsilon S^1(q, P, y_1, z_2, \varepsilon).$$

All other variables are defined in terms of (q, P, y_1, z_2) and $S(q, P, y_1, z_2)$:

$$\begin{cases} Q = q + \varepsilon\dfrac{\partial S^1(q, P, y_1, z_2, \varepsilon)}{\partial P}, & z_1 = y_1 + \varepsilon\dfrac{\partial S^1(q, P, y_1, z_2, \varepsilon)}{\partial z_2}, \\ p = P + \varepsilon\dfrac{\partial S^1(q, P, y_1, z_2, \varepsilon)}{\partial q}, & y_2 = z_2 + \varepsilon\dfrac{\partial S^1(q, P, y_1, z_2, \varepsilon)}{\partial y_1}. \end{cases} \tag{19}$$

Substitute (19) to (16) and (18), and equate Hamiltonians in "mixed" new-oldcoordinates :

$$\mathcal{H}_i\left(\left(\frac{q^2+\left(P+\varepsilon\frac{\partial S^1}{\partial q}\right)^2}{2}\right), y_1, z_2+\varepsilon\frac{\partial S^1}{\partial y_1}, \varepsilon\right) + \varepsilon g_i\left(q, P+\varepsilon\frac{\partial S^1}{\partial q}, y_1, z_2+\varepsilon\frac{\partial S^1}{\partial y_1}, \varepsilon\right) =$$

$$\mathcal{H}_i\left(\left(\frac{\left(q+\varepsilon\frac{\partial S^1}{\partial P}\right)^2+P^2}{2}\right), y_1+\varepsilon\frac{\partial S^1}{\partial z_2}, z_2, \varepsilon\right) + \varepsilon\bar{g}_i\left(\frac{\left(q+\varepsilon\frac{\partial S^1}{\partial P}\right)^2+P^2}{2}, y_1+\varepsilon\frac{\partial S^1}{\partial z_2}, z_2, \varepsilon\right) +$$

$$+\varepsilon g_{i+1}\left(q+\varepsilon\frac{\partial S^1}{\partial P}, P, y_1+\varepsilon\frac{\partial S^1}{\partial z_2}, z_2, \varepsilon\right).$$

Expand them in Taylor series, write out the terms of the same degree of ε and take into account, that $\partial\mathcal{H}_i/\partial y_1 = \partial\mathcal{H}_i/\partial z_2 = O(\varepsilon)$:

$$\mathcal{H}_i\left(\frac{q^2+P^2}{2}, y_1, z_2, \varepsilon\right) = \mathcal{H}_i\left(\frac{q^2+P^2}{2}, y_1, z_2, \varepsilon\right), \tag{20}$$

$$\frac{\partial\mathcal{H}_i}{\partial P}\frac{\partial S^1}{\partial q} - \frac{\partial\mathcal{H}_i}{\partial q}\frac{\partial S^1}{\partial P} + g_i(q, P, y_1, z_2, \varepsilon) = \bar{g}_i\left(\frac{q^2+P^2}{2}, y_1, z_2, \varepsilon\right). \tag{21}$$

Equation (21) is integrated with Lemma 1: $S(q, P, y_1, z_2, \varepsilon) = W(q, P, \mu)$, where $\mu = (y_1, z_2, \varepsilon)$. Function S is analytic function of all its variables and small parameter ε, so we can solve system (19), using Lemma 2. The solution is continuous and has continuous derivative, so it defines change of variables. We find necessary canonical transformation (17) and after substitution into (16) we can obtain explicit expression for $\mathcal{H}_{i+1}$ and εg_{i+1}.

4.2 Estimations.

Let r steps have been done. Domain D_i, where our Hamiltonian is considered after i steps, defined as $D_{i+1} = D_1 - 2(i-1)K\varepsilon$, where $D_1 = D + \frac{3}{4}\delta$, K -positive constant, defined below.

At first step we can easily show, that if $(q, p, y_1, y_2) \in D_1$ and ε is quite small, then (17) is defined and the following conditions are realized:

$$|g_1| + |\nabla g_1| < k_1\varepsilon, \qquad \left|Q^1 + P^1 + Z_1^1 + Z_2^1\right| < k_2, \qquad |\mathcal{H}_1 - \mathcal{H}_0| < k_3\varepsilon.$$

$\{k_i\}$ are positive constants. Really, from (18) we immediately obtain that $g_1 = O(\varepsilon^2)$. Using estimation procedure defined below all other inequalities are easily derived.

Let us accept inductive hypothesis that with $i : 1 \le i \le r$ following estimations hold:

$$|\mathcal{H}_i| < 2C, \qquad c < \left|\frac{\partial\mathcal{H}_i}{\partial I}\right| < 2C, \qquad |\nabla g_i| + |g_i| < M_i, \qquad M_i = \frac{k_1\varepsilon}{2^{i-1}}, \tag{22}$$

where $I = (q^2+p^2)/2$. Now one should find ε_1 and K such that if $0 < \varepsilon < \varepsilon_1$, $(Q, P, z_1, z_2) \in D_{r+1} = D_r - 2K\varepsilon$, the system (17) is defined with $i = r$ and (22) is satisfied with $i = r+1$.

For $\frac{\partial S^1}{\partial\varphi}$ we have the following estimations from definition of $\tilde{g}_i$ (7), equation (21) and form of solution (8) :

$$\left|\frac{\partial S^1}{\partial\varphi}\right| < |\tilde{g}_i| \le |g_i| < M_i.$$

The estimations for $\frac{\partial S^1}{\partial J}$, $J = (q^2+P^2)/2$ is obtained by the following way :

$$\frac{\partial S^1}{\partial J} = \frac{\partial}{\partial J}\left(\frac{1}{\frac{\partial\mathcal{H}_i}{\partial J}}\left(\frac{1}{2}\int\limits_0^\varphi \tilde{g}(\psi, J)d\psi + \frac{1}{2}\int\limits_\pi^\varphi \tilde{g}(\psi, J)d\psi\right)\right) =$$

$$= -\frac{\partial^2\mathcal{H}_i}{\partial J^2}\frac{S^1}{\frac{\partial\mathcal{H}_i}{\partial J}} + \frac{1}{\frac{\partial\mathcal{H}_i}{\partial J}}\frac{\partial}{\partial J}\left(\frac{1}{2}\int\limits_0^\varphi \tilde{g}(\psi, J)d\psi + \frac{1}{2}\int\limits_\pi^\varphi \tilde{g}(\psi, J)d\psi\right). \tag{23}$$

We can estimate $\frac{\partial^2 \mathcal{H}_i}{\partial J^2}$ by :

$$\left|\frac{\partial^2 \mathcal{H}_i}{\partial J^2}\right| < \left|\frac{\partial^2 \mathcal{H}_0}{\partial J^2}\right| + \varepsilon \left|\frac{\partial^2 \bar{g}_0}{\partial J^2}\right| + \varepsilon \sum_{j=1}^{i} \left|\frac{\partial^2 \bar{g}_j}{\partial J^2}\right|.$$

Using initial data we can state that $|\frac{\partial^2 \mathcal{H}_0}{\partial J^2}| < m_1$ and $|\frac{\partial^2 \bar{g}_0}{\partial J^2}| < k_4$, where $\{m_i\}$ are positive constant and doesn't depend on step i. For $|\frac{\partial^2 \bar{g}_j}{\partial J^2}|$ we use Cauchy estimations for analytic functions [4] :

$$\left|\frac{\partial^2 \bar{g}_i}{\partial J^2}\right| < \frac{M_i}{K\varepsilon}, \qquad \sum_{j=1}^{i} \left|\frac{\partial^2 \bar{g}_i}{\partial J^2}\right| < \sum_{j=1}^{i} \frac{k_1 \varepsilon}{2^{j-1} K \varepsilon} < \frac{2k_1}{K}.$$

Summing up, we obtain:

$$\left|\frac{\partial^2 \mathcal{H}_i}{\partial J^2}\right| < m_1 + k_4 \varepsilon + \frac{2k_1}{K}\varepsilon < k_5. \tag{24}$$

For $\frac{\partial g_i}{\partial J}$ we have the following estimations from (22):

$$\left|\frac{\partial g_i}{\partial J}\right| = \left|\frac{\partial \rho}{\partial J}\right| \left|\frac{\partial g_i}{\partial \rho}\right| < \frac{1}{\rho}\left(\left|\frac{\partial g_i}{\partial P}\right| + \left|\frac{\partial g_i}{\partial q}\right|\right) < \frac{1}{\rho} 2M_i,$$

where $\rho^2 = J$.

Then integral in (23) can be estimated as:

$$\left|\frac{1}{\frac{\partial \mathcal{H}_i}{\partial J}}\left(\frac{1}{2}\int_0^{\varphi} \frac{\partial \tilde{g}(\psi, J)}{\partial J} d\psi + \frac{1}{2}\int_{\pi}^{\varphi} \frac{\partial \tilde{g}(\psi, J)}{\partial J} d\psi\right)\right| < m_2 (\frac{4\pi M_i}{\rho}) < \frac{m_3 M_i}{\rho}.$$

Then expression (23) is estimated:

$$\left|\frac{\partial S^1}{\partial J}\right| < (k_5 + \frac{m_3}{\rho}) M_i.$$

Now we have the estimations for $\frac{\partial S^1}{\partial \varphi}$ and $\frac{\partial S^1}{\partial J}$ and can obtain estimations for $\frac{\partial S^1}{\partial q}$ and $\frac{\partial S^1}{\partial P}$.

$$\begin{aligned}\left|\frac{\partial S^1}{\partial P}\right| &= \left|\frac{\partial J}{\partial P}\frac{\partial S^1}{\partial J} + \frac{\partial \varphi}{\partial P}\frac{\partial S^1}{\partial \varphi}\right| < \rho(k_5 + \frac{m_3}{\rho}) M_i + \left|\frac{q}{q^2+P^2}\right| |g_i| < \\ &< (\rho k_5 + m_3) M_i + \left|\frac{q}{q^2+P^2}\right| \left|\frac{\partial g_i}{\partial q} q + \frac{\partial g_i}{\partial P} P\right| < \\ &< (\rho k_5 + m_3) M_i + 2M_i < m_4 M_i. \end{aligned} \tag{25}$$

For $\frac{\partial S^1}{\partial q}$ estimations obtain the same way.

$$\left|\frac{\partial S^1}{\partial q}\right| < m_5 M_i. \tag{26}$$

Estimations for the $\frac{\partial S^1}{\partial y_1}$ and $\frac{\partial S^1}{\partial z_2}$ are obtained easily by differentiation of $\tilde{g}_i$ in the definition of solution for S :

$$\left|\frac{\partial S^1}{\partial y_1}\right| < m_6 M_i, \qquad \left|\frac{\partial S^1}{\partial z_2}\right| < m_7 M_i. \tag{27}$$

Now we can define $k_6 = \max(m_4, m_5, m_6, m_7)$. Function g_{i+1} can be defined by :

$$|g_{i+1}| = \left|\frac{\varepsilon}{2}\left(\frac{\partial^2 \mathcal{H}_i}{\partial P^2}\right)_\theta \left(\frac{\partial S^1}{\partial q}\right)^2 - \frac{\varepsilon}{2}\left(\frac{\partial^2 \mathcal{H}_i}{\partial q^2}\right)_\theta \left(\frac{\partial S^1}{\partial P}\right)^2 + \left(\frac{\partial \mathcal{H}_i}{\partial z_2}\right)_\theta \frac{\partial S^1}{\partial y_1} - \left(\frac{\partial \mathcal{H}_i}{\partial y_1}\right)_\theta \frac{\partial S^1}{\partial z_2} + \right.$$
$$\left. +\varepsilon \left(\frac{\partial g_i}{\partial P}\right)_\theta \frac{\partial S^1}{\partial q} - \varepsilon \left(\frac{\partial \bar{g}_i}{\partial q}\right)_\theta \frac{\partial S^1}{\partial P} + \varepsilon \left(\frac{\partial g_i}{\partial z_2}\right)_\theta \frac{\partial S^1}{\partial y_1} - \varepsilon \left(\frac{\partial \bar{g}_i}{\partial y_1}\right)_\theta \frac{\partial S^1}{\partial z_2}\right|,$$

where $()_\theta$ means the derivative in the middle point. Thus, g_{i+1} estimates using (22), (24)–(27) :

$$|g_{i+1}| < \left|2\frac{\varepsilon k_5 k_6 M_i^2}{2} + 2\varepsilon\left(k_4 + \frac{2k_1}{K}\right) M_i + 4\varepsilon k_6 M_i^2\right| < k_7 M_i.\varepsilon$$

Using Cauchy estimations [4] we obtain estimations for ∇g_{i+1}:

$$|\nabla g_{i+1}| < \left|4\frac{\varepsilon k_5 k_6 M_i^2}{2K\varepsilon} + 2\varepsilon\left(k_4 + \frac{2k_1}{K}\right)\frac{M_i}{K\varepsilon} + 8\varepsilon k_6 \frac{M_i^2}{K\varepsilon}\right| < k_8 M_i \left(\varepsilon + \frac{1}{K}\right).$$

Choosing K quite large and ε quite small we obtain $k_7\varepsilon < \frac{1}{4}$, $k_8\left(\varepsilon + \frac{1}{K}\right) < \frac{1}{4}$. Then

$$|g_{i+1}| + |\nabla g_{i+1}| < \frac{M_i}{2} = M_{i+1},$$

and other inductive inequalities (22) are valid for $i = r + 1$. Therefore, we can do necessary changes of variables with chosen K, ε till D_r is not empty. After $r = (\frac{\delta}{4K\varepsilon}) > k_9/\varepsilon$ changes we have obtained:

$$|g_r| + |\nabla g_r| < \frac{k_1\varepsilon}{2^{r-1}} < c_2 exp(-\frac{1}{c_1\varepsilon}),$$

and other inequalities (4) will be satisfied.

Theorem is proved.

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References

[1] N. N. Bogolujbov, Yu. A. Mitropolski, Asymptotic methods in oscillation theory. ”Nauka”, Moscow, 1974; English transl. Problems of the asymptotic theory of nonstationary vibrations Jerusalem : Israel program for scientific translations, 1965 .

[2] A.I. Neishtadt, The separation of motions in systems with rapidly rotating phase. //Prikladnaya Mat. i Mechanika v.48, no. 2,1984, pp.197 -205.(Mat. Mekh. 48 (1984), no. 2, 197–204)

[3] J. Brüning, S. Yu. Dobrokhotov, Global Semiclassical Description of the Spectrum of 2-D Schrödinger Operators with Periodic Electric and Strong Magnetic Fields, to appear.

[4] V. I. Arnold, Small denominators and the problems of stability for classical and celestial mechanics, Uspehi Matematicheskih Nauk, 1963, vol. 18, part 6, c.91-192.

[5] V. I. Arnold, Mathematical methods of classical mechanics,§52.d, Nauka, Moscow, 1974.